

ON THE PARABOLIC, HYPERBOLIC AND DISCRETE FORMULATION OF THE HEAT CONDUCTION EQUATION

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NOMENCLATURE

c ,	mean speed of a molecule:
F ,	function defined by equation (8):
h ,	spatial increment, of the order of the mean free path:
i ,	spatial index:
I_1 ,	modified Bessel Function, 1st order:
k ,	time index:
l ,	wall thickness:
n ,	integer:
p ,	Laplace variable:
q ,	$\sqrt{(p^2 + 2p)}$:
t ,	time:
T ,	temperature:
u ,	unit step function, $u(t - x) = 1$ for $t > x$; $= 0$ for $t < x$:
v ,	speed of heat propagation:
x ,	Cartesian coordinate:
α ,	thermal diffusivity:
θ ,	transformed temperature:
λ ,	mean free path:
τ ,	propagation time increment $= h/v$, also dummy variable.

Subscripts and superscripts

0 ,	initial:
w ,	wall surface:
$*$,	dimensionless.

INTRODUCTION

THE PARABOLIC heat-conduction equation leads to solutions exhibiting infinite propagation speed of heat. This led many investigators to seek a modified form which will not permit infinite speed of propagation. As a result of these efforts it has been largely agreed that the damped wave equation, or the telegraph equation, is the true conduction equation [1-6]:

$$\frac{1}{v^2} \frac{\partial^2 T}{\partial t^2} + \frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T. \quad (1)$$

In this paper we present a solution for a thin layer subject to a step change of temperature on its both sides. The

results show that using the hyperbolic equation, the transient temperature rise may exceed the temperature of the boundaries as well as the initial temperature of the layer.

Although one may argue that thermodynamically (second law considerations) this solution is acceptable [6] it still seems to be unrealistic for gaseous material and possibly for most solids [7].

Thus we tend to conclude that for most cases equation (1) is also at most an approximation which is not valid for short periods of time, and in this sense it is not much better than the conventional diffusion equation.

A "difference" equation of conduction is suggested here based on ideas similar to the problem of "random walk" as described in [3]. The "difference" equation seems to satisfy the requirements expected of heat conduction, at least for gases, namely finite propagation speed as well as heat flow only from points of high temperatures towards points of lower temperature.

HYPERBOLIC SOLUTION

Consider a wall with uniform initial temperature T_0 . At time $t = 0$ both surfaces at $x = 0$ and $x = l$ are impulsively stepped to a temperature T_w . The solution of the diffusion parabolic equation to this problem is well documented and can be found, for example, in [8].

We focus our attention now to the solution of the diffusion hyperbolic equation to this problem, namely equation (1), with the following boundary and initial conditions:

$$T(x, 0) = T_0 \quad \text{and} \quad \frac{\partial T}{\partial t}(x, 0) = 0 \quad \text{for} \quad t = 0 \quad (2)$$

$$T(0, t) = T(l, t) = T_w \quad \text{for} \quad t > 0. \quad (3)$$

Dimensionless variables are defined

$$T^* = \frac{T(x, t) - T_0}{T_w - T_0}; \quad t^* = \frac{v^2 t}{2\alpha}; \quad x^* = \frac{vx}{2\alpha}. \quad (4)$$

Thus equation (1) takes the form

$$\frac{\partial^2 T^*}{\partial t^{*2}} + 2 \frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}. \quad (5)$$

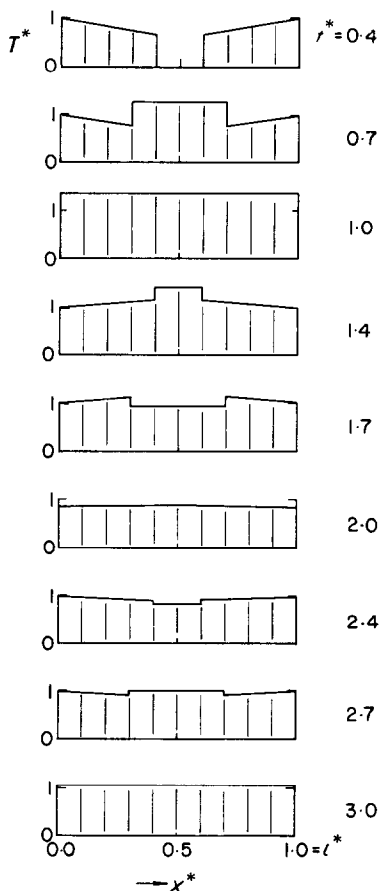


Fig. 1. Temperature distribution for hyperbolic solution.

The solution is obtained using Laplace transformation. The transformed solution $\theta(x^*, p)$ can be written in the form:

$$\theta(x^*, p) = \frac{1}{p} \sum_{n=1}^{\infty} \left[\exp\{-q[l^*(2n-2) + x^*]\} + \exp\{-q[l^*(2n-1) - x^*]\} + \exp\{-q[l^*(2n) - x^*]\} \right] \quad (6)$$

where $q = \sqrt{p^2 + 2p}$.

In this particular form equation (6) can be inverted term by term using [4].

The solution is given as follows:

$$T^*(x^*, t^*) = \sum_{n=1}^{\infty} \{F[l^*(2n-2) + x^*, t^*] + F[l^*(2n-1) - x^*, t^*] - F[l^*(2n-1) + x^*, t^*] - F[l^*(2n) - x^*, t^*]\} \quad (7)$$

where

$$F(x^*, t^*) = u(t^* - x^*) \times \left[e^{-x^*} + x^* \int_{x^*}^{t^*} e^{-\tau} \frac{I_1[\sqrt{(\tau^2 - x^{*2})}]}{\sqrt{(\tau^2 - x^{*2})}} d\tau \right]. \quad (8)$$

The result of equation (7) are given in Fig. 1 for the case where $l^* = 1$. The temperature profile $T^*(x^*, t^*)$ is plotted vs. x^* for $t^* = 0.4, 0.7, 1.0, 1.4, 1.7, 2.0, 2.4, 2.7$ and 3.0 .

The results clearly exhibit a damped wave character. When both wave fronts meet, a superposition of the left wave and the right wave (see for $t^* = 0.7$) results high temperature in the middle region of the wall. In particular, the temperature profile for $t^* = 1$ is interesting. In this case the temperature in the wall is uniformly above the surface temperatures (except for $x^* = 0$ and $x^* = l^*$ where $T^* = 1$). Though this phenomenon is classically admitted for propagating waves it is hard to believe that this can occur in a temperature field, namely that we can obtain for any period of time an almost uniform wall temperature which is higher than the imposed surface temperature T_w .

A possible explanation of this dilemma is given in the next section where we suggest a discrete formulation of the heat conduction equation.

THE DISCRETE FORMULATION

We divide a media into cells of a characteristic length h of the order of a mean free path of the molecules. For solids and liquids this length can be understood as a representative mean free path of the thermal excitation [9]. We focus our attention now on a one dimensional case. Designating the thermal propagation speed as v , the time for propagation to a distance h is $\tau = h/v$. The temperature field is represented discretely as $T_{ik} = T(ih, k\tau)$ and we write the heat conduction equation as

$$T_{i,k+1} = \frac{1}{2}(T_{i+1,k} + T_{i-1,k}). \quad (9)$$

The reasoning for accepting equation (9) is based on the idea of the random walk [3] and can be expressed directly as follows: The increment i is of the order of a mean free path away from the increments $i+1$ and $i-1$. Therefore molecules from the aforementioned increments will reach the interval i during a period τ while the original molecules at station i will escape during this period. This will result in region i after time τ , a temperature which is the mean of T_{i+1} and T_{i-1} .

Equation (9) clearly solves the paradox of the infinite speed of heat propagation. No response will be felt inside a semi infinite body, owing to a temperature change at the surface, at positions where $k > i$. On the other hand equation (9) also does not permit the overshoot of temperature described in the previous section. The temperature inside the body will always be bounded by the maximum and minimum initial and boundary temperatures.

Since h and τ are small, one can image an analytic continuous function $T(x, t)$ and expand T in a Taylor series on both sides of equation (9). The resulting equation yields

$$\begin{aligned} \frac{\partial T}{\partial t} \tau + \frac{1}{2} \frac{\partial^2 T}{\partial t^2} \tau^2 + \frac{1}{3!} \frac{\partial^3 T}{\partial t^3} \tau^3 + \dots \\ = \frac{1}{2} \left[\frac{\partial^2 T}{\partial x^2} h^2 + \frac{2}{4!} \frac{\partial^4 T}{\partial x^4} h^4 + \frac{2}{6!} \frac{\partial^6 T}{\partial x^6} h^6 + \dots \right], \quad (10) \end{aligned}$$

Retaining the first term on the left-hand side and the first term on the right-hand side leads to the classical parabolic equation where $\alpha = h^2/2\tau$.

Taking two terms in the left-hand side leads to the hyperbolic formulation, equation (1), where, as expected $v = h/\tau$.

Obviously more differential equations can be written by using more terms of equation (10). Alternatively h and τ may be calculated with respect α and v as follows

$$h = \frac{2\alpha}{v}, \quad \tau = \frac{2\alpha}{v^2}. \quad (11)$$

We may now check our previous assumptions as to the values of h and τ . Taking the value of α for gases as $\alpha \cong \frac{1}{3} c\lambda$ we obtain that

$$h \cong \frac{2c}{3v} \lambda, \quad \tau \cong \frac{2c\lambda}{3v^2}. \quad (12)$$

Thus indeed, since the propagation speed of heat v is of the order of the average molecule speed c , h is indeed of the order of the mean free path λ . Also τ is of the order of the collision time. This is consistent with our previous assumptions.

On the basis of accepting the discrete equation (9) as the more exact formulation of heat conduction, it is clear that both the parabolic and hyperbolic equations are just first and second order differential approximations. As a limiting approximation both equations are subject to a nonphysical description of certain phenomena, like the well known infinite propagation speed and the temperature overshoot phenomenon described previously.

At this point it is worthwhile to mention that S. Goldstein [5] proposed the random walk model with correlation for the diffusion process. In this manner Goldstein managed to obtain a finite propagation speed for diffusion while allowing h to approach zero. In our view this is not necessary and we just have to accept the finiteness of h . In fact, accepting Goldstein's model will lead to results of the previous section. Also the results of the solution presented in [4] should be treated cautiously. This solution is based on a strict solution of the hyperbolic equations. As shown here this may lead to some physical distortion.

SUMMARY AND CONCLUSIONS

The hyperbolic heat conduction equation was introduced by many authors in order to tackle with the infinite propagation speed of heat as predicted by the conventional parabolic equation. The present paper demonstrates that the hyperbolic equation may also lead into physically doubtful solutions of a physical problem.

The "difference" heat conduction equation takes into account the "grainy" structure of matter and tackles successfully the aforementioned physical difficulties that appear in the hyperbolic equation as well as the parabolic one. It clearly shows that both hyperbolic and parabolic differential equations can be derived through a truncated Taylor series expansion.

We suggest that a discrete temperature field in time and space is closer to physical reality than the continuous description. The spatial interval h is of the order of the mean free path of the thermal excitation, and the time increment τ equals h/v . As a result we suggest extreme caution in interpreting results obtained through the differential equations (of both kinds) pertaining to detailed temperature field description for spatial distance of the order h and time of the order τ .

Though the analysis presented here is based on simplified considerations, and concerning primarily gaseous material, our general conclusions probably hold true for most liquids and solids.

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